Final exam Calculus-3 (10 points free):

Total points to obtain 100 [*In all problems provide a brief justification for what you do*]

<u>Problem 1</u> (15 points)

Consider the series: $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$ (<u>a: 10 points</u>) For which values of x is the series convergent?
(<u>b: 5 points</u>) Find the sum of the series

<u>Problem 2</u> (10 points)

Show that: $\frac{1+x}{1-x} = 1 + 2\sum_{n=1}^{\infty} x^n \quad \text{if } |x|<1$

Problem 3 (20 points)

Suppose a mass m is attached to a spring with spring constant k, and let $k = m\omega^2$. If an external force $F(t) = F_o \cos(\omega_o t)\ (\omega_o \neq \omega)$ is applied, then we have the equation of motion for non-zero dissipation (c>0):

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \quad (1)$$

Show that a particular solution of Eq. (1) $(c^2 - 4mk < 0)$ is given by: $x_p(t) = \left\{ \frac{F_o}{\sqrt{(k - m\omega_0^2)^2 + (c\omega_0)^2}} \right\} \cos(\omega_0 t + \delta)$

$$\tan\delta = -c\omega_0/(k - m\omega_0^2)$$

Tip: $x(t) = c_1\cos(\omega t) + c_2\sin(\omega t) = A\cos(\omega t + \delta)$ with $A = \sqrt{c_2^{\,2} + c_1^{\,2}}$ and $\tan\delta = -c_2/c_1$

Problem 4 (10 points) Assume a function *f(x)* to have Fourier transform: $F(k)=\int_{-\infty}^{+\infty} f(x)e^{-i2\pi kx}dx$

Consider also the Fourier transform definition of the Dirac Delta function: $\delta(k)=\int_{-\infty}^{+\infty} e^{-i2\pi kx}dx$

Then derive the Fourier Transform of the function: $f(x)=\sin[4\pi k_o x]$

Problem 5 (15 points)

Find the sine Fourier series solution to the differential equation $\frac{d^2y}{dx^2}+ky=f(x)$ with *k* an integer, *f(x)* an odd function [*f(x)=-f(-x)*] , and the boundary conditions *y(0)=y(4)=0*

Tip : The Fourier sine expansion has the general form $Y(x)=\sum_{n=1}^{\infty} b_n \sin(n\pi x/L),\ b_n=(2/L)\int_0^L Y(x)\sin(n\pi x/L)dx,$ for $x\in[0,L]$

Problem 6 (20 points)

Consider the boundary value problem for the one-dimensional heat equation for a bar with the zero-temperature ends:

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2},\quad u=u(x,t),\ \ t>0,\ \ 0<x<L,$$
$$u(x,0)=f(x),$$
$$u(0,t)=0,\quad u(L,t)=0,\ \ t\geq 0.$$

u(x,t): Temperature

(a: 10 points) Show that the general solution for *u(x,t)* is given by:

$$u(x,t)=\sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t}\sin\frac{n\pi}{L}x\ ,\quad \lambda_n=\frac{cn\pi}{L}$$

(b: 10 points) Derive the solution *u(x,t)* for the case *f(x)=50sin(πx/40) + 30sin(3 πx/40)* (with L=40)